Real roots of random polynomials: universality close to accumulation points

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Abstract
We identify the scaling region of a width $O(n^{-1})$ in the vicinity of the
accumulation points $t = \pm 1$ of the real roots of a random Kac-like polynomial
of large degree n. We argue that the density of the real roots in this region tends
to a universal form shared by all polynomials with independent, identically
distributed coefficients c_i, as long as the second moment $\sigma = \mathbf{E}(c_i^2)$ is finite.
In particular, we reveal a gradual (in contrast to the previously reported abrupt)
and quite nontrivial suppression of the number of real roots for coefficients
with a nonzero mean value $\mu_n = \mathbf{E}(c_i)$ scaled as $\mu_n \sim n^{-1/2}$.

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1. Introduction

Recently, there was an essential resurrection of interest in statistical properties of zeros of
random polynomials and random analytic functions [1–15].

The distribution of real roots for random polynomials was studied in classical papers by
Bloch and Polya [16], Littlewood and Offord [17], Kac [18] and many others. Many results
and an overview of the field are given in the recent monograph by Farahmand [22].

As is well known, for the case of algebraic polynomials

$$f_n(t) = c_0 + c_1 t + \cdots + c_{n-1} f^{n-1} \tag{1.1}$$

where the real coefficients c_i are independently and identically distributed around mean zero,
the roots are mostly located near ± 1. A natural intuitive picture for such an accumulation was
proposed by Kac. Indeed, if all coefficients c_i are of the same order then only for $|t| = 1$ will
the terms have a good chance to interact and cancel each other producing zero.

In particular, for Gaussian-distributed coefficients with mean $E[c_j] = 0$ and variance
$E[c_j^2] = \sigma$, the mean density of real zeros of an algebraic polynomial $p_n(t)$ is given by the
Kac formula

$$p_n(t) = \frac{\sqrt{A_n(t)C_n(t) - B_n^2(t)}}{\pi A_n(t)} \tag{1.2}$$

where

$$A_n(t) = \sigma \sum_{j=0}^{n-1} t^{2j} = \sigma \frac{t^{2n} - 1}{t - 1} \qquad B_n(t) = \frac{1}{2}\frac{d}{dt}A_n(t)$$

$$C_n(t) = \frac{1}{4}\frac{d^2}{dt^2}A_n(t) + \frac{1}{4t}\frac{d}{dt}A_n(t).$$

Usually, one is interested in the limit of polynomials of high degree n. For a fixed t the density is given by:

$$\lim_{n\to\infty} p_n^{(f)}(t \neq \pm 1) = p^{(f)}(t) = \frac{1}{\pi|1 - t^2|} \tag{1.3}$$

and

$$p_n(\pm 1) = \frac{1}{\pi}\left[\frac{n^{2-1}}{12}\right]^{1/2}.$$

The expected number of real zeros is known to have the following asymptotics:

$$N_n(-\infty, \infty) = \int_{-\infty}^{\infty} p_n(t)\,dt = \frac{2}{\pi}\log n + C + O(n^{-2}) \tag{1.4}$$

where the leading logarithmic term was found by Kac himself, and the corrections, in particular, the constant

$$C = \frac{2}{\pi}\left\{\int_0^1 \sqrt{\frac{1}{v^2} - \frac{1}{\sinh^2 v}}\,dv - \int_1^\infty\left[\frac{1}{t} - \sqrt{\frac{1}{v^2} - \frac{1}{\sinh^2 v}}\right]dv\right\} \tag{1.5}$$

was found by Wilkins [20]. In fact, the asymptotic result $N_n(-\infty, \infty) = \frac{2}{\pi}\log n$ was proved to be very universal, i.e. insensitive to the details of the distribution of coefficients c_i (see discussion and further references in [22]). A similar universality result is also available for the variance of the total number of real zeros [21].

The nature of the informal arguments by Kac suggests that close to the values $t = \pm 1$ many random terms of comparable magnitude add up to form the polynomial, and the emerging object should be universal in the spirit of the central limit theorem.

Another natural question arises about the expected number of real zeros for coefficients with a nonvanishing mean value $\mathbf{E}[c_i] = \mu$. It is straightforward to include the mean value to the standard derivation of the Kac formula for gaussian-distributed coefficients and arrive at the following expression:

$$p_n^\mu(t) = p_n(t)\,e^{-\frac{1}{2}[\Sigma_1(t) + \Sigma_2(t)]}\int_0^\infty dq\, q\cosh(q\sqrt{\Sigma_1(t)})\,e^{-\frac{q^2}{2}} \tag{1.6}$$

$$\Sigma_1(t) = \frac{A_n(t)D_n^2(t)}{A_n(t)C_n(t) - B_n^2(t)} \qquad \Sigma_2(t) = \frac{G_n^2(t)}{A_n(t)} \tag{1.7}$$

where

$$D_n(t) = -G_n(t)\frac{B_n(t)}{A_n(t)} + \frac{d}{dt}G_n(t) \qquad G_n(t) = \mu\sum_{j=0}^{n-1} t^j = \mu\frac{t^n - 1}{t - 1} \tag{1.8}$$

and $P_n(t)$, $A_n(t)$, $B_n(t)$ and $C_n(t)$ are defined in (1.2).

Earlier investigation [22] has revealed that any, whatever small, mean value $\mu > 0$ asymptotically, for large n, kills exactly half of real zeros and converts the above quoted result 1.4 for their number into $N_n(-\infty, \infty) = \frac{1}{\pi}\log n$. It is this surprising abruptness that motivated us to have a closer look at the origin of such a behaviour. This question turned out to be also related to exploring the appropriately scaled vicinity of the accumulation points.

2. Scaling and universality

It is quite clear (and can be verified numerically) that the root density function $p_n(t)$ as well as its limiting form (1.3) are both not universal. On the other hand, the essence of the Kac argument discussed above and the mentioned universality of $N_n(-\infty, \infty)$ suggests that universal features have to emerge in close vicinity of the accumulation points $t = \pm 1$. In fact, it is easy to understand that the relevant vicinity of $|t| = 1$ is of the order of n^{-1} for large degree n. We will refer to such domain as the '*local scaling regime*', as opposed to keeping the distance $|1 \pm t|$ from the accumulation points $t = \pm 1$ fixed in the limit $n \to \infty$, the latter regime being referred to as the '*global*' one. In particular, our main goal is to verify proposition 2.1.

Proposition 2.1. *Let coefficients $\{c_j\}_{j=0}^{n-1}$ of a random polynomial $f_n(t)$ be i.i.d real random variables with $\mathbf{E}[c_j] = \mu_n > 0$ and finite variance chosen to be unity: $\mathbf{E}[c_j^2] = 1$. Let $N_n(x_1, x_2)$ be the number of real zeros of $f_n(t)$ in the interval $\left[1 - \frac{x_2}{n}, 1 - \frac{x_1}{n}\right]$, with $0 \leqslant x_1 < x_2 < n$. Then*

$$\lim_{n\to\infty} \mathbf{E}[N_n(x_1, x_2)] = \int_{x_1}^{x_2} p_\alpha(v)\,\mathrm{d}v \tag{2.9}$$

where $\alpha = \lim_{n\to\infty}\left(n\mu_n^2\right)$ and the density function $p_\alpha(v)$ is given by

$$p_\alpha(v) = \frac{1}{\pi}\sqrt{\frac{1}{v^2} - \frac{1}{\sinh^2 v}}\,\mathrm{e}^{-\alpha J(v)}\int_0^\infty \mathrm{d}q\, q \cosh\left(q\sqrt{\alpha M(v)}\right)\mathrm{e}^{-\frac{q^2}{2}} \tag{2.10}$$

$$J(v) = \frac{\sinh^2(v/2)}{(v/2)^2}\frac{1}{1 + \sinh v/v} \qquad M(v) = \frac{\sinh v/v}{\cosh^2(v/2)}\frac{1 - \sinh v/v}{1 + \sinh v/v}. \tag{2.11}$$

A similar statement is valid for the vicinity of second accumulation point $t = -1$ for $\mu_n < 0$.

Here we present explicit arguments in favour of the validity of such a proposition for the simplest case $\mu_n = 0$. The 'local regime' formula (2.10) reduces in that case to the following simple expression:

$$p_0(v) = \frac{1}{2\pi}\sqrt{\frac{1}{v^2} - \frac{1}{\sinh^2 v}} \tag{2.12}$$

The modifications required to include nonzero mean μ_n are self-evident and left to the reader.

Our starting formula is the following representation for the number of real roots $N_n(x_1, x_2)$ of $f_n(t)$ in the interval $\left[1 - \frac{x_2}{n}, 1 - \frac{x_1}{n}\right]$ (see e.g. [6, 22]):

$$N_n(x_1, x_2) = \int_{1-\frac{x_2}{n}}^{1-\frac{x_1}{n}} \mathrm{d}t \int_{-\infty}^\infty \mathrm{d}y\, |y|\mathcal{D}_n(0, y; t) \tag{2.13}$$

where

$$\mathcal{D}_n(x, y; t) = \mathbf{E}\left[\delta\left(x - \sum_{j=0}^{n-1} c_j t^j\right)\delta\left(y - \sum_{j=0}^{n-1} jc_j t^{j-1}\right)\right] \tag{2.14}$$

stands for the joint probability density of the random polynomial $f_n(t)$ and its derivative over t. Let us introduce in the above representation the scaling variables $v, \tilde{x}$ and $\tilde{y}$ by relations:

$$t = 1 - v/n \qquad x = n^{1/2}\tilde{x} \qquad y = n^{3/2}\tilde{y}.$$

Changing the variables of integration one finds:

$$N_n(x_1, x_2) = \int_{x_1}^{x_2} \mathrm{d}v \int_{-\infty}^{\infty} \mathrm{d}\tilde{y} |\tilde{y}| \tilde{\mathcal{D}}_n(0, \tilde{y}; v) \tag{2.15}$$

where

$$\tilde{\mathcal{D}}_n(\tilde{x}, \tilde{y}; v) = \mathbf{E}\left[\delta \left(\tilde{x} - \frac{1}{n^{1/2}} \sum_{j=0}^{n-1} c_j (1 - v/n)^j \right) \delta \left(\tilde{y} - \frac{1}{n^{3/2}} \sum_{j=0}^{n-1} j c_j (1 - v/n)^{j-1} \right) \right]. \tag{2.16}$$

Evidently each of the scaled random variables $\tilde{x}$ and $\tilde{y}$ is a sum of n independent, although not identically distributed terms, with the magnitude of fluctuations of each term depending on the summation index j. Calculating the variances of the scaled variables we find that they are given by

$$A_n(v) = \mathbf{E}[\tilde{x}^2] = \frac{1}{n} \sum_{j=0}^{n-1} (1 - v/n)^{2j} = \frac{1 - (1 - v/n)^{2n}}{v(2 - v/n)} \tag{2.17}$$

and

$$B_n(v) = \mathbf{E}[\tilde{y}^2] = \frac{1}{n^2} \sum_{j=0}^{n-1} j(1 - v/n)^{2j-1} = \frac{1}{2n^2} \frac{\mathrm{d}}{\mathrm{d}a} \left(\frac{a^n - 1}{a - 1} \right)_{a=(1-v/n)^2} \tag{2.18}$$

and their covariance is just

$$C_n(v) = \mathbf{E}[\tilde{x}\tilde{y}] = \frac{1}{n^3} \sum_{j=0}^{n-1} j^2 (1 - v/n)^{2j-2} = \frac{1}{n^3} \frac{\mathrm{d}}{\mathrm{d}a} \left[a \frac{\mathrm{d}}{\mathrm{d}a} \left(\frac{a^n - 1}{a - 1} \right) \right]_{a=(1-v/n)^2}. \tag{2.19}$$

We further note that all three quantities $A_n(v)$, $B_n(v)$ and $C_n(v)$ have a well-defined finite large n limit:

$$A_n(v) \to A_\infty = \frac{1 - \mathrm{e}^{-2v}}{2v} \qquad B_n(v) \to B_\infty = -\frac{1}{2} \frac{\mathrm{d}A_\infty}{\mathrm{d}v} \qquad C_n(v) \to C_\infty = \frac{1}{4} \frac{\mathrm{d}^2 A_\infty}{\mathrm{d}v^2} \tag{2.20}$$

coinciding with the limiting values of A, B, C in equation (1.2) after rescaling $t = 1 - v/n$ and the limit $n \to \infty$ for a fixed v. Now invoking the local central limit theorem [26] we infer that the limiting joint probability density $\tilde{D}_\infty$ of the scaled variables $\tilde{x}$ and $\tilde{y}$ tends when $n \to \infty$ to the normal law with variances A_∞, B_∞ and covariance C_∞, as if the polynomial coefficients were normal with unit variance. Therefore, the Kac formula for the density of real roots should be asymptotically valid in this case, and the formula (2.12) immediately follows after substituting those limiting values into the Kac expression (1.2). Separation of the two regimes and identification of the scaling is in some sense similar to 'local' versus 'global' scaling regimes in spectra of random matrices. Only when the spectral parameters are scaled appropriately will the various correlation functions characterizing the eigenvalues of large random matrices show a surprisingly robust universality [24, 25].

Although the expressions (2.10), and especially equation (2.12) look very natural (cf the structure of the asymptotic result (1.4)) and their universality stems from such basic fact as the central limit theorem, we failed to trace a similar statement in the available literature on random polynomials with real coefficients. For complex zeros a kind of 'local' regime was studied in much detail by Shepp and Vanderbei [3], and Ibragimov and Zeitouni [8] who showed that those zeros tend to concentrate asymptotically on the unit circle. Only very
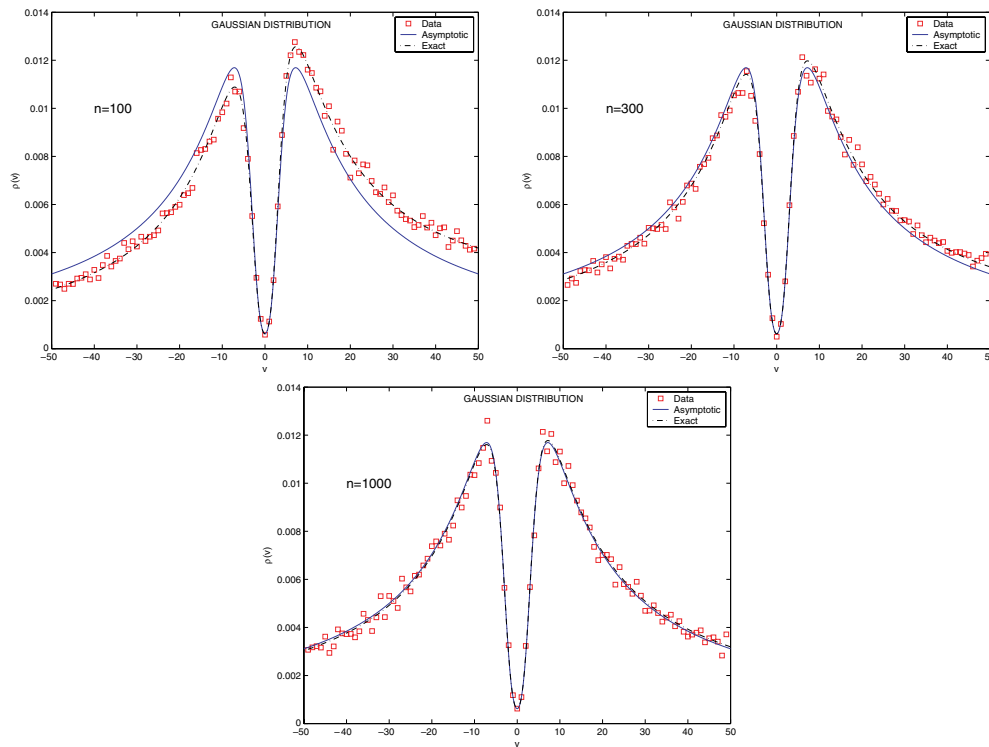
Figure 1. Numerical versus analytical density profiles for real roots of polynomials with gaussian-distributed coefficients. Each dataset corresponds to 100 000 realizations of random polynomials.

recently (squared) expression of the type (2.12) independently emerged in studies of complex zeros of polynomials with complex i.i.d. coefficients (see Shiffman and Zelditch, [15]). A different kind of universality results was addressed in a very recent paper [7] which appeared when the present paper was under completion.

3. Numerical examples, discussion and perspectives

It is interesting and informative to have a look at the profiles of the real root density in the '*local scaling regime*' around the point $t = 1$ (i.e $v = 0$) for some typical values of the parameters, and compare them with the results of numerical simulations. In figure 1 we plotted the density profiles obtained by a direct numerical search for real roots of polynomials with gaussian-distributed coefficients for degrees $n = 100, 300$ and 1000. The (scaled) mean value $\alpha_n = \mu_n^2 n$ of the coefficients was chosen to be $\alpha = 10$ and the variance was always kept unity. The results can be compared with both the exact predictions of the Kac-type formula (1.6) for the same values of n, σ and $\mu_n = \sqrt{\alpha/n}$, and with the asymptotic universal profile of (2.10). It is trivial to see that as n increases the exact Kac-like expression begins to coincide with the asymptotic profile, and the analytical curve agrees well with the numerical data. To verify universality we also performed numerical simulations for the case of coefficients uniformly distributed in the interval of the widths $1/\sqrt{3}$ around the same mean values μ_n. The picture looks very similar, see figure 2, and again agreement with the asymptotic formula for large values of n is very good.
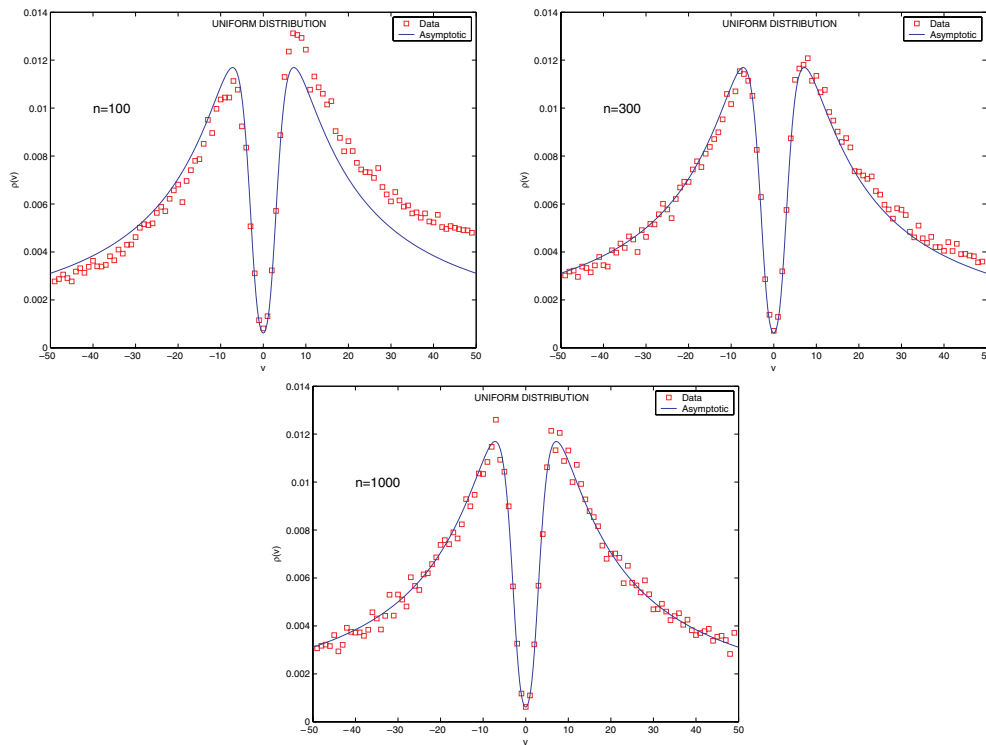
Figure 2. Numerical data versus asymptotic density profile (2.10) for the uniformly distributed coefficients. Each dataset corresponds to 100 000 realizations of random polynomials.

One can note a couple of interesting features. First simple observation is that the exact Kac formulae are asymmetric about $v = 0$ for 'small' n (in the '*local regime*') but it becomes 'asymptotically symmetric' with increasing n, as predicted by (2.10). This feature is easy to understand in view of the global inversion symmetry $t \to t^{-1}$ which holds exactly for all polynomials with i.i.d. coefficients. In the local regime close to accumulation points this symmetry implies the asymptotic reflection symmetry $v \to -v$.

The most surprising feature is a non-trivial double-peak structure of the density profile, and it deserves to be discussed in more detail.

In fact, it turns out that as α increases the shape of the density profile described by formula (2.10) changes from that with one maximum to that with two symmetric maxima as illustrated in figure 3.

This fact can be verified analytically by considering the small-v expansion:

$$p_\alpha(v) = \frac{1}{\pi\sqrt{3}} \, \mathrm{e}^{-\alpha/2} \left[1 - \frac{v^2}{2}\left(\frac{1}{5} - \frac{1}{6}\alpha\right) + \frac{v^4}{36}\left(\frac{137}{350} - \frac{5}{8}\alpha + \frac{1}{12}\alpha^2\right) + O(v^5) \right] \qquad (3.21)$$

valid for $\alpha v^2 \ll 1$. We see that for $\alpha > \alpha_c = 6/5$ the local maximum at $v = 0$ is converted to a minimum. On the other hand, for $|v| \gg \alpha$ we easily find from equation (2.10) the α-independent far tail decay law $p_\alpha(v) \approx 1/\pi|v|$ which ensures that at least two symmetric maxima have to appear at $v \neq 0$. Note that the far tail values are much larger for $\alpha \gg 1$ than the exponentially small density $p_\alpha(0) = \frac{1}{\pi\sqrt{3}} \, \mathrm{e}^{-\alpha/2}$ at the origin, and thus the maxima have to be quite pronounced.
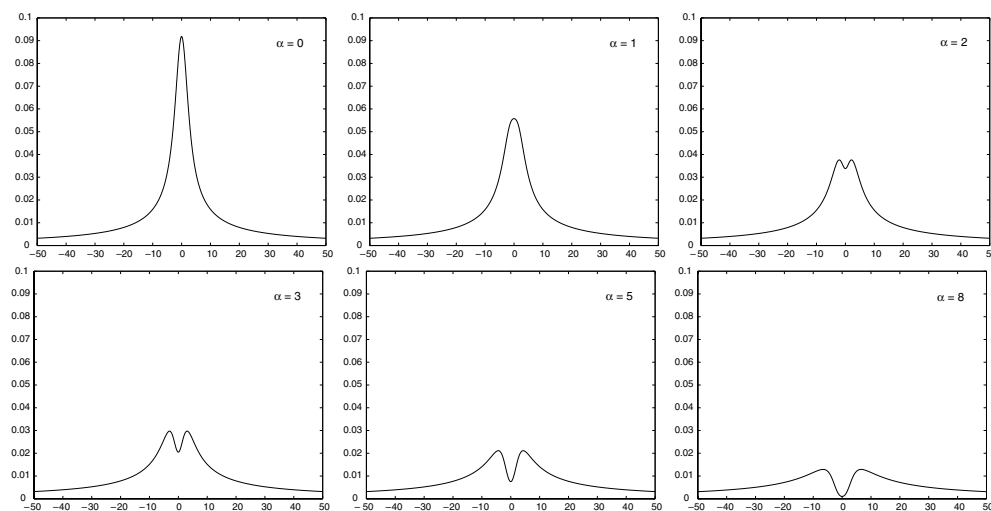
Figure 3. Suppression of the density profile by increasing scaled mean value α from 0 to 8 for a degree 100 polynomial.

The precise reason for such a nonmonotonous behaviour is not clear to us at the moment, and deserves further investigations. We consider that feature as an indication of a rather nontrivial statistics of zeros accumulating in the scaling region around the points ± 1.

Fixing the value of μ and performing the limit $n \to \infty$ amounts to letting $\alpha \to \infty$. This results in vanishing density of the real roots $p_\alpha(v) \to 0$ for any fixed v. This fact corresponds to observed suppression of exactly half of the real zeros (the vicinity of the second accumulation point is not at all affected by the nonvanishing mean μ_n). In contrast, correctly tuning the mean value with the parameter n by letting $\mu_n \sim n^{-1/2}$ and magnifying the vicinity of the accumulation point via the correct scaling $1 - t \sim n^{-1}$ results in a density function (2.10) and thus describes a gradual (as opposed to abrupt) suppression of real zeros in the scaling regime, (see figure 3).

Let us now briefly discuss open questions and perspectives for further research. A very interesting development of the theory of random polynomials comes from the paper by Bleher and Di [6]. The authors discovered nontrivial correlations between positions of real zeros of algebraic polynomials. The correlations can be conveniently characterized by correlation functions of the polynomial zeros:

$$K(t_1, \ldots, t_l) = \lim_{n \to \infty} \mathbf{E} \left[\lim_{\Delta_1, \ldots, \Delta_l \to 0} \frac{N^{(f)}(t_1, t + \Delta_1) \ldots N^{(f)}(t_l, t_l + \Delta_l)}{|\Delta_1| \ldots |\Delta_l|} \right] \qquad (3.22)$$

where $N^{(f)}(a, b)$ is the number of real zeros of the polynomial $f(t)$ in the interval $[a, b]$.

The consideration of correlation functions studied by Bleher and Di was restricted, in our terminology, to the 'global regime'. It should be possible to derive 'local regime' formulae for those quantities. The expressions are expected to be universal in the same sense as the 'local regime' formula for expected number of roots, equations (2.9), (2.10). The simplest nontrivial quantity of that type should be the variance $\text{var}[N_n(x_1, x_2)]$ of the number of real roots in the vicinity of $t = 1$.

Another natural object to study is the variation of the number of real polynomial roots against a small change of the vector of real coefficients $\mathbf{c} = c_0, \ldots, c_{n-1}$. More specifically, change $\mathbf{c} \to \mathbf{c}_v = \mathbf{c} + v\mathbf{b}$ assuming the components of the vector $\mathbf{b}$ to be i.i.d. standard

Gaussian. The parameter $v \geqslant 0$ is used here to control the magnitude of the perturbation. By using $f_v(t)$ to denote the perturbed polynomial an interesting question arises, that is can we generalize the correlation functions (3.22) to the following *parametric* correlation functions:

$$K_{v_1,\ldots,v_l}(t_1,\ldots,t_l) = \lim_{n \to \infty} \mathbf{E}\left[\lim_{\Delta_1,\ldots,\Delta_l \to 0} \frac{N^{(f_{v_1})}(t_1, t+\Delta_1)\ldots N^{(f_{v_l})}(t_l, t_l+\Delta_l)}{|\Delta_1|\ldots|\Delta_l|}\right] \qquad (2.23)$$

which reflects the change in the positions of the real roots. Similar objects are of interest in the theory of random matrices and disordered systems (see [27] and references therein).

As a preliminary step of our research we evaluated the simplest nontrivial parametric correlation function, following the paper [6] and found in the 'local regime':

$$\frac{K_{0,v}(t,t)}{[p(t)]^2} = 1 + \frac{1}{v}\arcsin\frac{1}{\sqrt{1+v^2}}.$$

It will be interesting to attack the problem in full generality, both for 'global' and especially for 'local' regime, where the results are expected to be universal.

In fact, our initial interest in the properties of random polynomials was stimulated by the fact that closely related methods can be applied to study the properties of irregular eigenfunctions in 'quantum billiards'. The eigenfunctions $\Psi(x, y)$ at the point with the coordinate vector $\vec{r} = (x, y)$ are solutions of the Helmholtz equation: $-\Delta\Psi(\vec{r}) = E\Psi(\vec{r}); \vec{r} \in \Omega$ where Ω is a connected compact domain with the boundary $\partial\Omega$, and Δ is the Laplacian.

Recently, Smilansky and collaborators [23] suggested looking for the eigenfunctions at the point $\vec{r}$ in the following representation:

$$\Psi(r, \theta) = \sum_{l=-L}^{L} a_l J_l(k|r|) \, e^{il\theta} \qquad (3.24)$$

where $J_l(x)$ stands for the Bessel functions, r, θ are polar coordinates of the observation point, and the integer L is given in terms of the wavenumber k and the perimeter D of the billiard boundary $\partial\Omega$ as $L = \frac{1}{2}\left[\frac{kD}{\pi}\right]$. The complex coefficients a_l satisfying $\overline{a_l} = (-1)^l a_{-l}$ are taken to be i.i.d. complex Gaussian variables with unit variance, in accordance to the Berry's conjecture. Based on such a formula the authors of [23] managed to calculate the mean number N and variance var(N) of the intersection of nodal lines with the billiard boundary $\partial\Omega$. For the Dirichlet boundary conditions along the boundary curve ∂D parametrized as $R(\theta)$, with $0 \leqslant \theta < 2\pi$, the problem turned out to be equivalent to counting the number of real zeros of the function

$$u(\theta) = \sum_{l=-L}^{L} a_l J_l'(kR(\theta)) \, e^{il\theta}$$

in the interval $\theta \in [0, 2\pi)$. Those developments provide an interesting possibility for applying ideas and methods from the theory of random polynomials to describe chaotic eigenfunctions.

Similar methods can be hopefully used to study sensitivity of the nodal lines of eigenfunctions of 'quantum billiards' with respect to the perturbation of the billiard parameters. For example, one may wish to study parametric variations of the quantities involved, with the role of the external parameter played by a slight random variation of the boundary curve $R(\theta)$ or by any other tunable physical parameter.

Acknowledgments

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